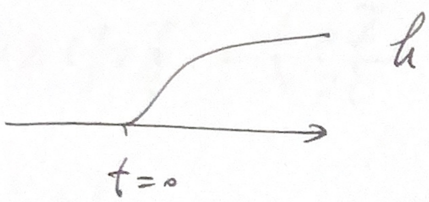
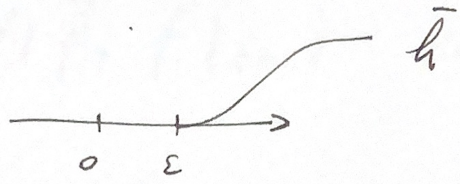
Feb 28

Last time: formal solution + local invertiblity of $\mathcal{E}$





$$(\bar{h}, \bar{f}_0) = \mathcal{E}(f_0)$$

+ small neighbourhood.

f smooth in space variable
only need smoothness in time variable.

convergence ?

$$\begin{cases} \frac{\partial f}{\partial t} = E(f) \\ f|_{t=0} = 0 \end{cases}$$

Here's how the induction works in more detail ...

at $t=0$ $\quad f \sim a_0(x) + a_1(x)\,t + a_2(x)\,t^2 + \cdots$

$$\frac{\partial f}{\partial t} = a_1(x) + 2a_2(x)\,t + \cdots$$

$$RHS \sim E(f_0) + \left.\frac{\partial E(f)}{\partial t}\right|_{t=0} \cdot t + \cdots$$

$$\frac{\partial E(f)}{\partial t} = DE(f_0)(a_1(x))$$

$$\Rightarrow \quad a_1(x) = E(f_0)$$

$$a_2(x) = DE(f_0) \cdot a_1(x)$$

$$\vdots$$

To check: tame condition $\longrightarrow$ convergence of the solution.

It remains to show

$$\mathcal{D}\mathcal{E}(f)\tilde{f} = \left(\frac{\partial \tilde{f}}{\partial t} - DE(f)\tilde{f},\ \tilde{f}|_{t=0}\right) \quad \text{invertible}$$

(I) $\begin{cases} \dfrac{\partial \tilde{f}}{\partial t} - DE(t)\tilde{f} = \tilde{h} & \text{has a uniq sol.} \\ \tilde{f} = \tilde{f}_0 \end{cases}$

(II) $\tilde{f} = \tilde{f}(\tilde{h}, \tilde{f}_0, f)$ is smooth tame.

To prove the (I).

define $\quad P(f) = DE(f) + L^*(f) \cdot L(f)$

<u>Claim</u> $\quad \dfrac{\partial \tilde{f}}{\partial t} = P(f)\tilde{f}$ is parabolic

pf. $\quad \sigma P(f)(\xi) = \sigma DE(f) + (\sigma(L(f)))^* + \sigma(L(f))$.

$$L(f) : C^\infty(X, F) \longrightarrow C^\infty(X, G)$$

(0,2) tensor in Ricci case

Pick v evalue of $\sigma(P(f))(\xi)$

Integrability cond. $\quad \sigma(L(f))(\xi) \cdot \sigma(DE(f))(\xi) = 0$

Then $\sigma L(f)\, \sigma(P(f))(\xi)\, v = \lambda\, \sigma L(f)\, v$

$= \sigma L(f) \cdot \underbrace{\sigma(DE)(f)\, v}_{=0}$

$+ \sigma(L(f))\, \sigma(L(f))^* \sigma(L(f))\, v$

$\Rightarrow \langle \sigma(L(f)) \cdot \sigma(L(f))^* \sigma L(f)\, v,\ \sigma L(f)\, v \rangle$

$= \lambda \langle \sigma L(f)\, v,\ \sigma L(f)\, v \rangle$

$= \lambda\, |\sigma L(f)\, v|^2 \longrightarrow \lambda = 0$ or $\lambda > 0$

$LHS = |\sigma(L(f))^* \sigma L(f)\, v|^2 = RHS.$

Recall $\sigma(DE(f)) : \mathrm{Ker}\, \sigma(L(f)) \longrightarrow \mathrm{Ker}\, \sigma(L(f))$ has all the evalues with strictly +ive real parts. (assumption)

When $\lambda = 0$, $LHS \to \sigma(L(f))^* \sigma(L(f))\, v = 0$

$\Rightarrow \langle \sigma(L(f))^* \sigma L(f)\, v,\ v \rangle = 0$

$\Rightarrow |\sigma L(f)\, v|^2 = 0 \qquad v \in \mathrm{Ker}\, \sigma(L(f)).$

By def of P, (the second term vanishes)

$\sigma\, DE(f)\, v = \sigma P(f)\, v = \lambda v = 0$

v is an evector of DE corresponds to $\lambda = 0$] contradiction

$\Rightarrow \lambda > 0$ and P is parabolic.

$$\frac{\partial \tilde{f}}{\partial t} - DE(f)\,\tilde{f} = \frac{\partial \tilde{f}}{\partial t} - P(f)\,\tilde{f} + L^*(f)\,\tilde{g} = \tilde{h}$$

where $\tilde{g} = L(f)\,\tilde{f}$.

Evolution equ of $\tilde{g}$:

$$\frac{\partial \tilde{g}}{\partial t} - M(f)\,\tilde{f} = \tilde{k}$$